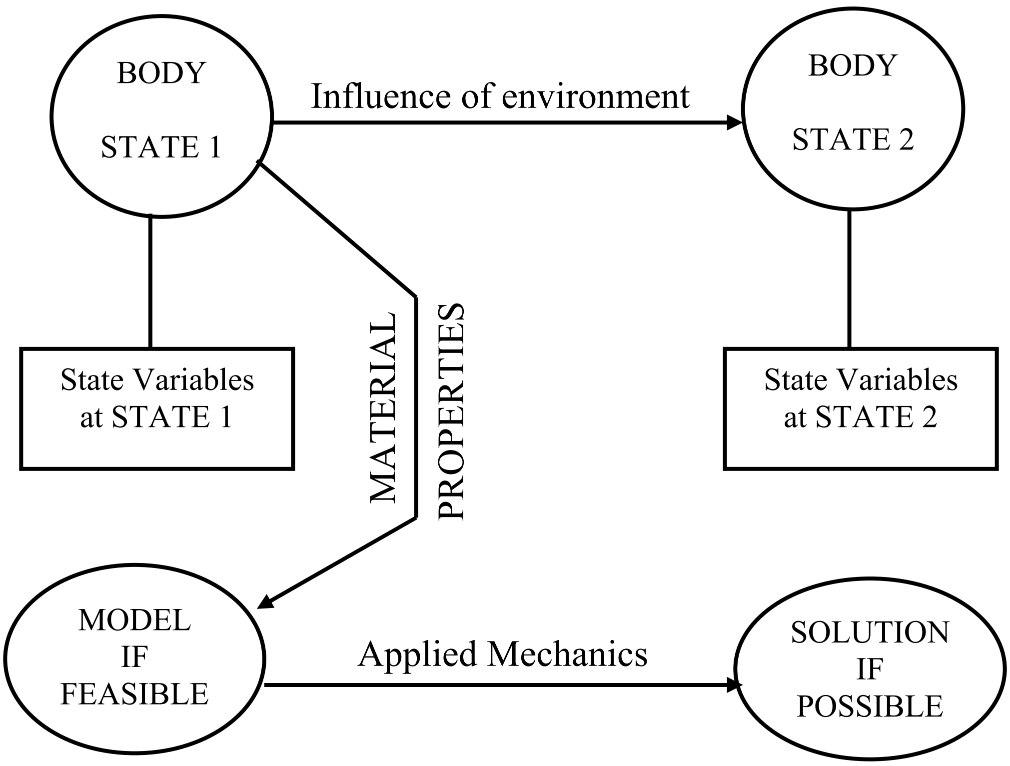
Chapter 1

Foundation of Solid Mechanics and Variational Methods

1- Some Fundamental Concepts

1-1- Physical Problems, Mathematical Models, Solutions
The main objective of this section is describing the concepts of body and mathematical modeling. Procedures for formulation and solution of a Mathematical model of a physical problem are discussed. The following diagram shows a general view of the modeling from body to model to solution.



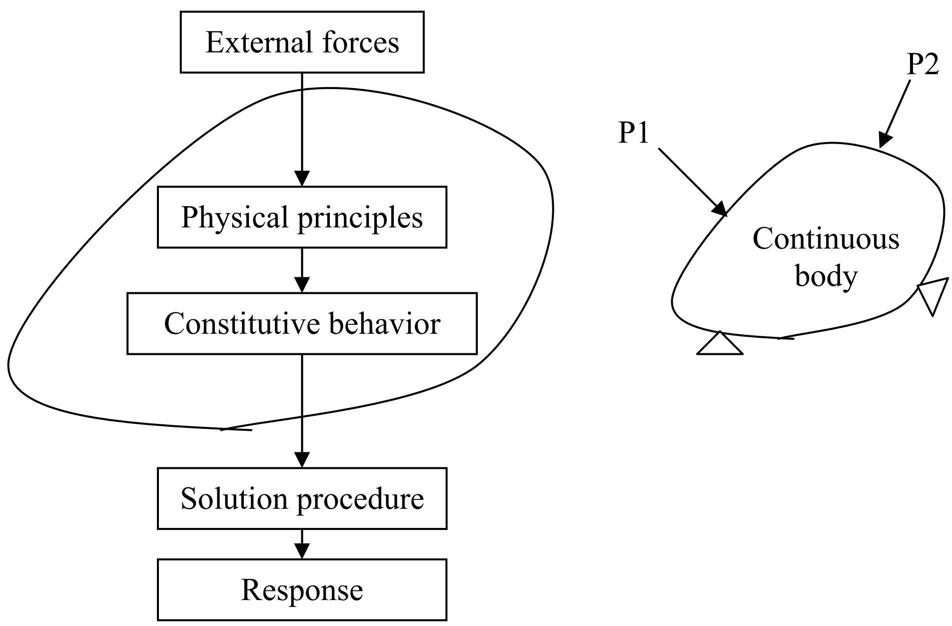
State variables involve displacements, velocities, pressure, temperature, stress, strain, charge, position, etc.
Influence of environment can be due to forces, temperature changes, etc.
Properties are determined from laboratory testing.

1-2- Continuum Mechanics
Things that we can perceive, see, hear, or build can be explained by using certain principles and laws of natures: conservation of mass, energy, linear

and angular momenta, the laws of electromagnetic flux, and the concept of thermodynamic irreversibility. These are among the fundamental principles on which the subject of mechanics is based.

The subject of continuum mechanics is based on the foregoing governing principles, which are independent of the internal constitution of material. However, the response of a system or a medium subjected to (external) forces can not be determined uniquely only with the governing field equations derived from the basic principles. The internal constitution of material plays an important role in the subject of continuum mechanics.

Study of the response of a substance or body under external excitation constitutes the major endeavor in engineering. In engineering applications, the response behavior can be studied at macroscopic level without considering atomic and molecular structure. The subject of studying material behavior at the macroscopic level can be called continuum mechanics.

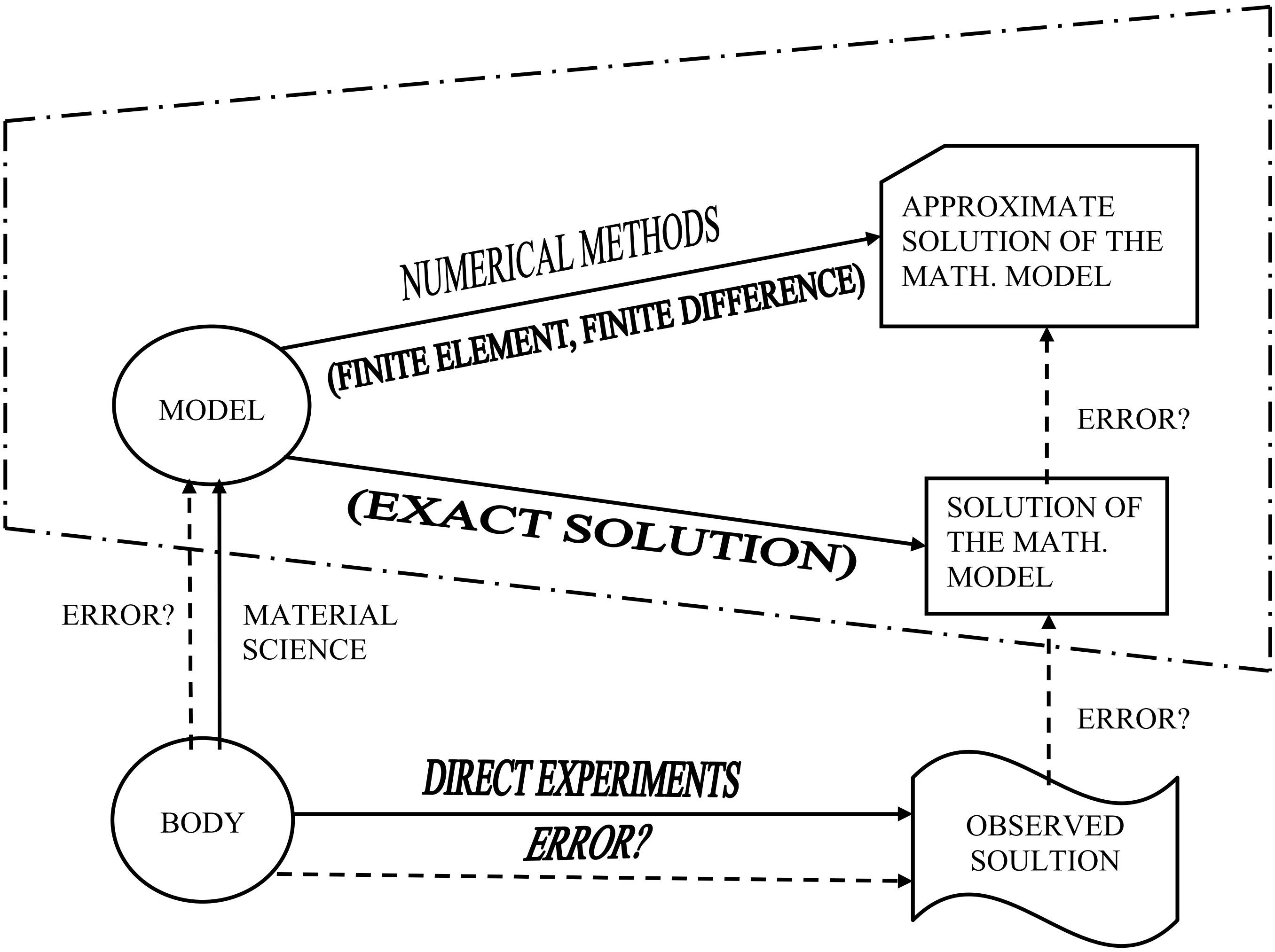


By invoking physical principles and constitutive behavior, we obtain equations governing the behavior of continuous system (a boundary value problem).
A solution to a boundary value problem in continuum mechanics requires constitutive equations in addition to the governing field equations. The basic principles governing Newtonian mechanics are a) conservation of mass, b) conservation of momentum, c) conservation of moment of

momentum (or angular momentum), d)conservation of energy, and e)laws of thermodynamics; these principles are considered to be valid for all materials irrespective of their internal constitution. Therefore, a unique solution to a boundary value problem in continuum mechanics cannot be obtained only with the application of governing field equations. Hence a unique determination of the response require additional consideration that account for the nature of different materials. The equations that model the behavior of a material are called ‘constitutive equations’ or ‘constitutive laws’ or ‘constitutive model’.

1-3- Boundary value problem solution

A solution to a BVP can be obtained using different approaches. The following diagram shows a schematic view of the problem.



Model studies or direct experiment include checking of the approximate solution with the state variables in laboratory which involves dimensional analysis and similitude.

1-4- Approximate solution of a boundary value problem

A mathematical model (BVP) of a real-life problem is often difficult to obtain an exact solution. The finite element method (FEM) can be viewed as a method of finding approximate solutions for the BVP problems.

Two approaches of Weighted Residual Method (WRM) and Energy Methods (EM) are used for finding approximate solution of BVP. A number of schemes are employed under the WRM, among which are collocation, subdomain, least squares, and Galerkin's methods. Galerkin's method has been the most commonly used residual method for finite element applications. This method is based on minimization of the residual left after an approximate or trial solution is substituted into the differential equation governing a problem. The EM procedures are based on the idea of finding consistent states of bodies or structures associated with stationary values of a scalar quantity assumed by the loaded bodies. In engineering, usually this quantity is a measure of energy or work. The process of finding stationary values of energy requires use of mathematical disciplines called variational calculus involving use of variational principles. For many problems, both approaches yield exactly the same results.

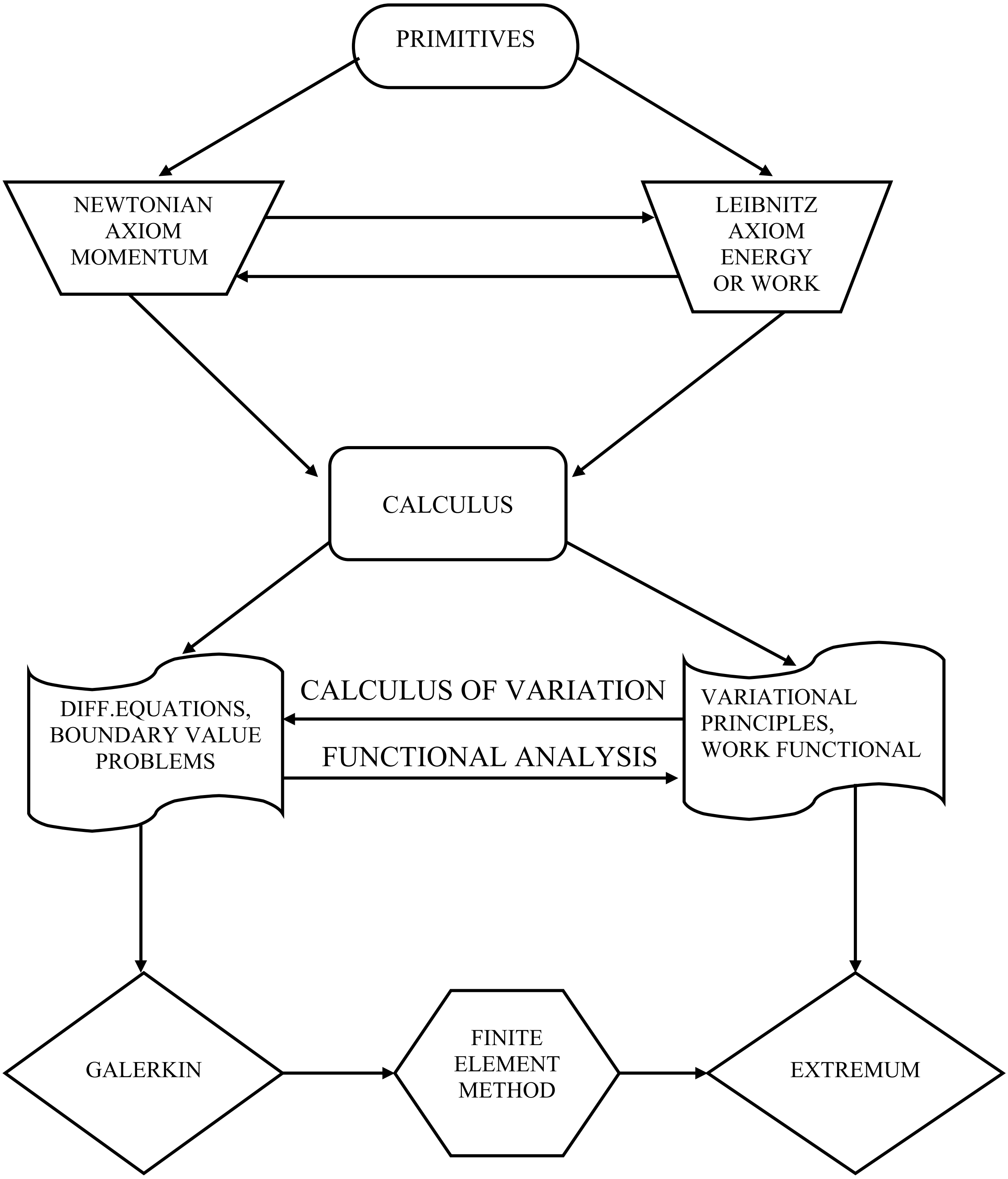
The following diagram shows a schematic view of use of these two approaches. Primitives are those involve physical quantities associated with the state variables, e.g. Time, Length, Force, etc. Based on the primitives we establish the axioms for problem solving, i.e. try to obtain a solution for the assumed model. Here the primary objective is to make sure that the mathematical model represents the real body.

Choice of axioms depends on the type of problem, form of geometry and the physical quantities involved. There are two kinds of axioms in applied mechanics.

i) Newtonian Axiom (Newton's Axiom)
   It defines force as momentum change, vector forces act on each particle of the body and an equilibrium differential equation (or momentum balance) governs the solution throughout the body.

ii) Leibnitz Axiom (Work Axiom)
   It defines work as the effect of forces acting on the body from which a work function is obtained, e.g. potential energy, complementary energy, kinetic energy, etc. A solution is

obtained as an extremum problem, e.g., a minimum or a maximum or a saddle point problem.



2- Concepts of Stress, Strain, Constitutive Relations and Various Form of Energy

Ref : Energy and Finite Element Methods in Structural Mechanics
By: I.H. Shames 1985

2-1- STRESS

2-1-1- Force Distributions

In study of continuous media 2 classes of forces exist:

a. body force distribution

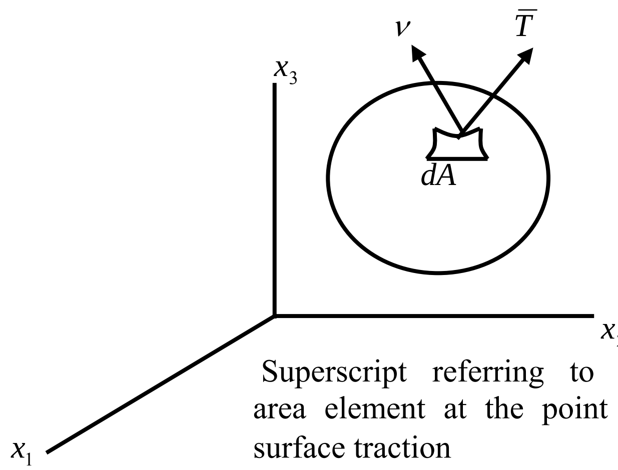
It acts directly on the distribution of matter in the domain of specification.

$$\frac{\overline{B}(x,y,z,t)}{\text{Vector notation}} \quad or \quad \frac{B_i(x_1,x_2,x_3,t)}{\text{Index notation}} \quad \begin{cases} \text{Per unit mass} \\ \text{Per unit volume} \end{cases}$$

b. Surface Tractions

In discussing a continuum there may be some boundary with force distributed on the boundary.
The force is applied to such boundary directly from material outside the domain.

$$\overline{T}(x,y,z,t) \;\; or \;\; T_i(x_1,x_2,x_3,t) \qquad \begin{cases} \text{Per unit area} \\ \text{need not be normal to the area element} \end{cases}$$



ν normal to dA

Superscript referring to the direction of the area element at the point of application of the surface traction

$$\overline{T}^{(\nu)}(x,y,z,t) \quad or \quad T_i^{(\nu)}(x_1,x_2,x_3,t)$$

If the area element has the unit normal in the x_j direction then We would express the traction vector on this element as:

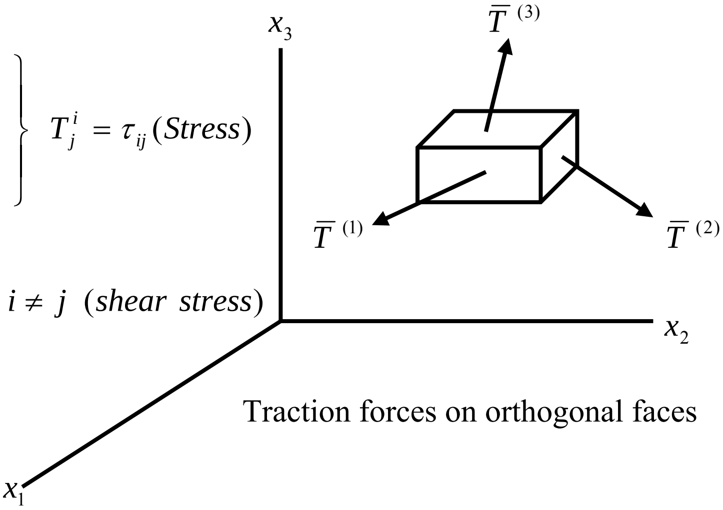
$$\overline{T}^{(j)} \quad or \quad T_i^{(j)} = T_{ji}$$

2-1-2- STRESS

$$\left.\begin{aligned} T_i^{(1)} &= \tau_{11}, \tau_{12}, \tau_{13} \\ T_i^{(2)} &= \tau_{21}, \tau_{22}, \tau_{23} \\ T_i^{(3)} &= \tau_{31}, \tau_{32}, \tau_{33} \end{aligned}\right\} \quad T_j^i = \tau_{ij}\,(Stress)$$

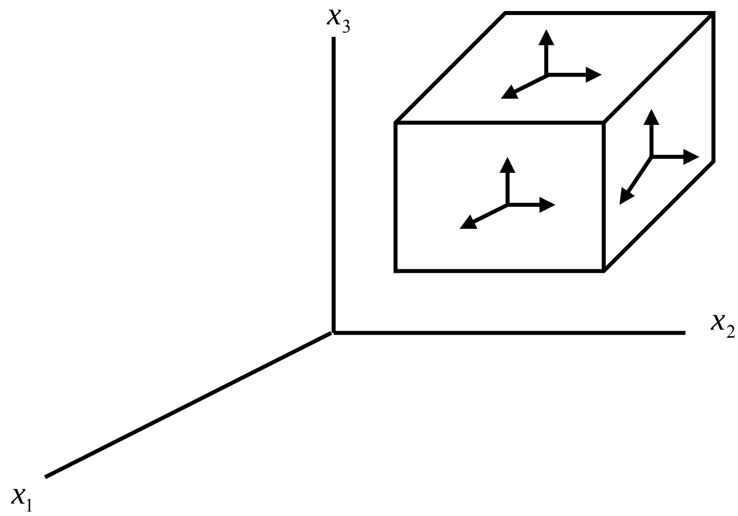
τ_{ii} = *normal stress*

τ_{ij} = *off diagonal terms* $i \neq j$ (*shear stress*)



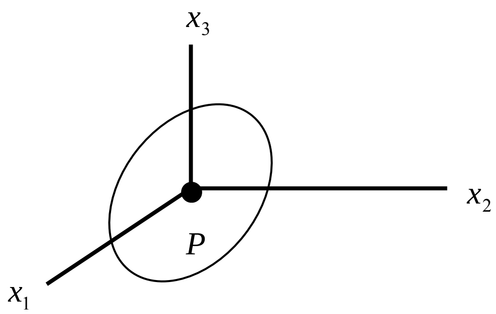
Traction forces on orthogonal faces

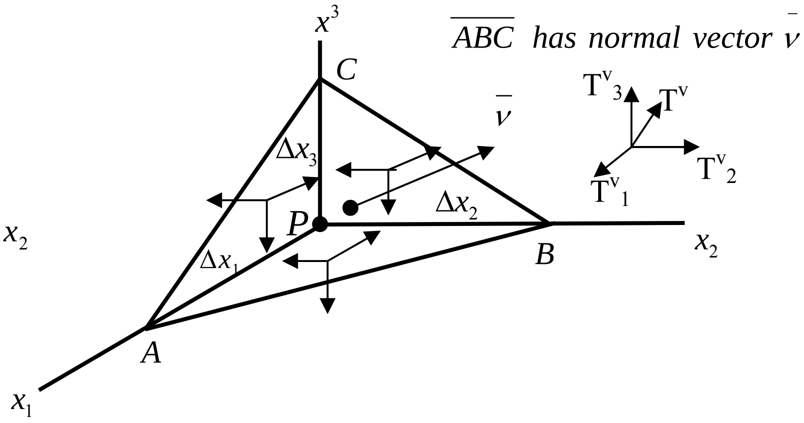
Sign convention: Normal stress directed outward from interface (+) tensile stress
Normal stress directed toward surface (-) compressive stress
Shear stress is (+) if both stress itself and unit normal point in +ive coordinates directions or both points in –ive coordinate directions.



Knowing T_{ij} for a set of axes, i.e. for three orthogonal interfaces at a point, we can determine a stress vector $\overline{T}^{(\nu)}$ for an interface at the point having any direction whatever.

Consider a point P in a continuum (any point in the domain)





Newton's law for the mass center in x_1 direction $\Longrightarrow$ <u>Cauchy's Formula</u>

$$T_i^{(\nu)} = T_{1i}\nu_1 + T_{2i}\nu_2 + T_{3i}\nu_3 \quad or \quad \overline{T}_i^{(\nu)} = T_{ji}\nu_j = T_{ij}\nu_j$$

Knowing T_{ij} we can get the traction vector for any interface at the point.
This formula can be used to relate tractions on the boundary to stresses directly next to the boundary.

Prove Cauchy's Formula.

2-1-3- Equations of Motion

Consider an element of the body of mass dm at P

Newton's 2nd law:

$$d\bar{f} = dm\ \dot{\bar{V}}$$

$$\oiint_S \bar{T}_i^{(\upsilon)} dA + \iiint_D \bar{B}_i\, dV = \iiint_D \dot{\bar{V}}_i\, \rho dV$$

$\bar{T}_i^{(\upsilon)} \rightarrow \tau_{ij}\nu_j$

Gauss' Theorem

Consider a continuous, differentiable n[th] order tensor field $T_{jk....}$ over a volume V with its boundary surface defined by S. The Gauss's Theorem in a generalized form is given by:

$$\iiint_V \frac{\partial T_{jk}\cdots}{\partial x_i}\, \mathrm{dV} = \iint_S (T_{jk}\ldots)\nu_i\, dA$$

or:

$$\iiint_V (T_{jk\ldots})_{,i}\, \mathrm{dV} = \oiint_S (T_{jk}\ldots)\nu_i\, dA$$

where ν_i are the direction cosines of the unit outward normal. For $T_{jk....}$ a zero order Tensor, say a scalar function φ,

$$\iiint_V \varphi_{,i}\, \mathrm{dV} = \oiint_S \varphi\nu_i\, dA$$

or:

$$\iiint_V \bar{\nabla}\varphi \mathrm{dV} = \oiint_S \varphi\, d\bar{A}$$

where the differential area $d\bar{A} = \bar{\nu}\, dA$. The above equation is generally referred to as Gauss' Law.

$$\iiint_D (\tau_{ij,j} + \bar{B}_i - \dot{\bar{V}}_i\rho)\, dV = 0$$

D is arbitrary $\Longrightarrow$ $\boxed{\tau_{ij,j} + B_i = \rho\dot{V}_i}$

Using moment of momentum equation will also result in:

$\bar{M} = \dot{\bar{H}}_0 \quad \Rightarrow \quad \tau_{jk} = \tau_{kj}$

Further investigations:

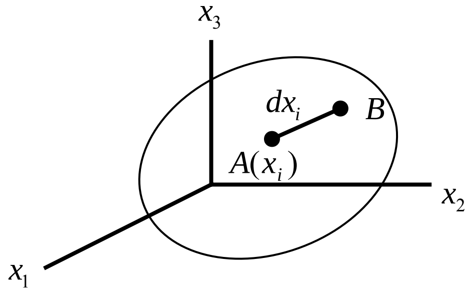
- Transformation Equations for stress
- Principal stresses (given a system of stresses for an orthogonal set of interfaces at a point , we can associate a stress vector for interfaces having any direction in space

$$T_i^V = \tau_{ij}\nu_j$$

Now: is there a direction ν such that stress vector is collinear with ν?

2-2- STRAIN

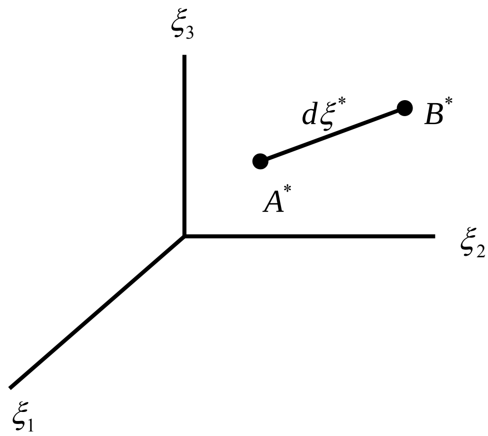
Means of expressing the deformation of a body



Line segment in the undeformed geometry
If body is given a rigid body motion ⟹ each line segment in the body under goes no change in length.
Change in length of line segments in the body, (or distance between points) can serve as a measure of deformation (change of shape and size) of the body.

$$(\overline{AB})^2 = ds^2 = dx_i dx_i \qquad \text{distance between points}$$

When forces are applied, body will deform. It is convenient now to consider that the x_i reference is labeled the ξ_i reference when considering deformed state:



Deformation can be depicted by mapping of each point from coordinate x_i to coordinate ξ_i. We can say then for a deformation:

$\xi_i = \xi_i (x_1, x_2, x_3)$

Since mapping is one-to-one:

$x_i = x_i (\xi_1, \xi_2, \xi_3)$:

We can express:

$$dx_i = \left(\frac{\partial x_i}{\partial \xi_j}\right) d\xi_j \qquad\qquad d\xi_i = \left(\frac{\partial \xi_i}{\partial x_j}\right) dx_j$$

$$ds^2 = dx_i dx_i = \frac{\partial x_i}{\partial \xi_m} \frac{\partial x_i}{\partial \xi_k} d\xi_m \, d\xi_k$$

$$\overline{A^* B^*}^{\,2} = ds^{*2} = d\xi_i \, d\xi_i = \frac{\partial \xi_i}{\partial x_k} \frac{\partial \xi_i}{\partial x_l} dx_k \, dx_l$$

$$ds^{*^2} - ds^2 = \left(\frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_k}{\partial x_j} - \delta_{ij} \right) dx_i \, dx_j = 2\varepsilon_{ij} \, dx_i \, dx_j$$

$$ds^{*^2} - ds^2 = \left(\delta_{ij} - \frac{\partial x_k}{\partial \xi_i} \frac{\partial x_k}{\partial \xi_j} \right) d\xi_i \, d\xi_j = 2\eta_{ij} \, d\xi_i \, d\xi_j$$

Strain terms:

Green strain

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_k}{\partial x_j} - \delta_{ij} \right)$$

Lagrange coordinates (ε_{ij} expressed as function of coordinates in the undeformed state)

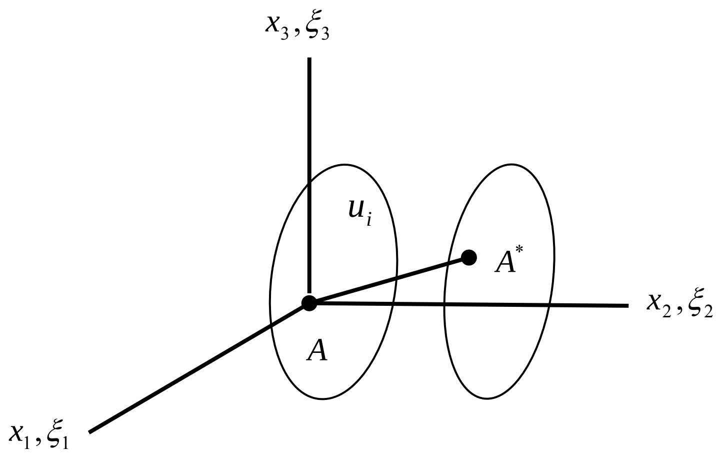
Almansi measure of strain

$$\eta_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial x_k}{\partial \xi_i} \frac{\partial x_k}{\partial \xi_j} \right)$$

Eulerian coordintes (η_{ij} formulated as function of coordinate for deformed state)

displacement field u_i

$u_i = \xi_i - x_i$



We may express u_i as a function of Lagrange coordinate x_i, in which case it expresses the displacement from the position x_i in the undeformed state to the deformed position ξ_i.

On the other hands u_i can equally well be expressed in terms of ξ_i, the Eulerian coordinates; in which case it expresses the displacement that must have taken place to get to the position ξ_i from some undeformed configuration.

$$\left.\begin{aligned} \frac{\partial x_i}{\partial \xi_j} &= \delta_{ij} - \frac{\partial u_i}{\partial \xi_j} \\ \frac{\partial \xi_i}{\partial x_j} &= \frac{\partial u_i}{\partial x_j} + \delta_{ij} \end{aligned}\right\} \quad \text{substitution in previous equation for } \varepsilon_{ij}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) \longrightarrow \text{Initial undeformed geometry}$$

Indicate what must occur during a given deformation.

$$\eta_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial \xi_j} + \frac{\partial u_j}{\partial \xi_i} - \frac{\partial u_k}{\partial \xi_i}\frac{\partial u_k}{\partial \xi_j}\right) \longrightarrow \text{Deformed instantaneous geometry of body}$$

Indicate what must have occurred to reach this geometry from an earlier undeformed state.

So far no restriction on magnitude of deformation,
Infinitesimal strain:

$$\frac{\partial u_i}{\partial x_j} << 1 \qquad \frac{\partial u_i}{\partial \xi_j} << 1$$

$$\frac{\partial J(x_i)}{\partial \xi_i} = \frac{\partial J}{\partial x_j}\left(\frac{\partial x_j}{\partial \xi_i}\right) = \frac{\partial J}{\partial x_j}\left[\frac{\partial}{\partial \xi_i}\left(\xi_j - u_j\right)\right] = \left(\delta_{ij} - \frac{\partial u_j}{\partial \xi_i}\right)\frac{\partial J}{\partial x_j}$$

for infinitesimal strain $\frac{\partial u_j}{\partial \xi_i}$ can be dropped $\longrightarrow$ $\boxed{\frac{\partial}{\partial \xi_i} = \frac{\partial}{\partial x_i}}$

$\Longrightarrow$ no need to distinguish between Eulerian and lograngian coordinates in expressing strains

$$\varepsilon_{ij} = \eta_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

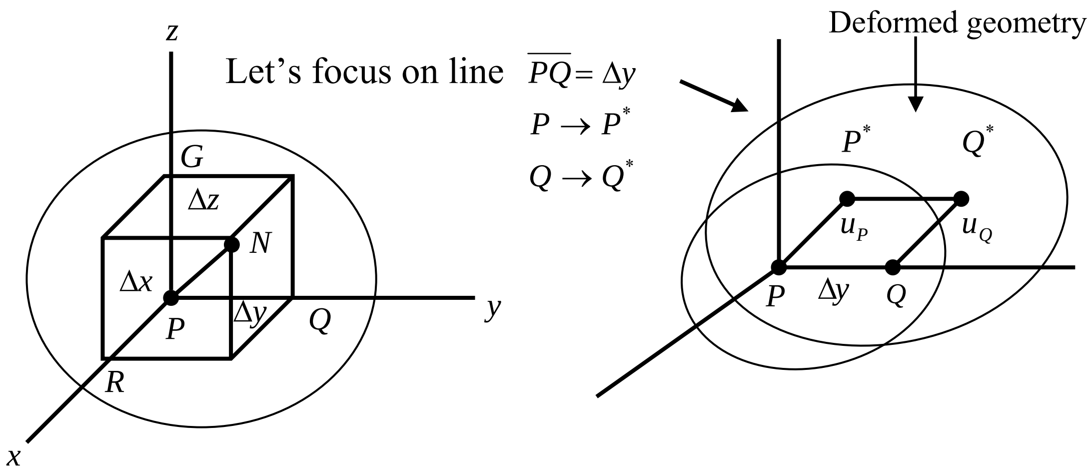
$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \qquad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{1}{2}\gamma_{xy} \qquad \gamma_{ij} = \text{engineering shear strain}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = \frac{1}{2}\gamma_{yz}$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \qquad \varepsilon_{xx} = \frac{1}{2}\left(\frac{\partial u_u}{\partial z} + \frac{\partial u_z}{\partial x}\right) = \frac{1}{5}\gamma_{nz}$$

2-2-1- Physical interpretation of strain terms

A small rectangular parallelepiped at P.
We have also placed a Cartesian reference at P. Imagine the body has some deformation:



Projection of $\overline{P^*Q^*}$ in the y direction $(\overline{P^*Q^*})_y$

$$(\overline{P^*Q^*})_y = \Delta y + (u_y)_Q - (u_y)_P$$

Taylor series for $(u_y)_Q$ in terms of $(u_y)_P$:

$$= \Delta y + \left[(u_y)_p + \left(\frac{\partial u_y}{\partial y}\right)_P \Delta y + \ldots\right] - (u_y)_p$$

$$= \Delta y + \left(\frac{\partial u_y}{\partial y}\right)_P \Delta y + \ldots$$

Net y component of elongation of segment $\overline{PQ}$

$$(\overline{P^*Q^*})_y - \Delta y = \left(\frac{\partial u_y}{\partial y}\right)_P \Delta y + \ldots$$

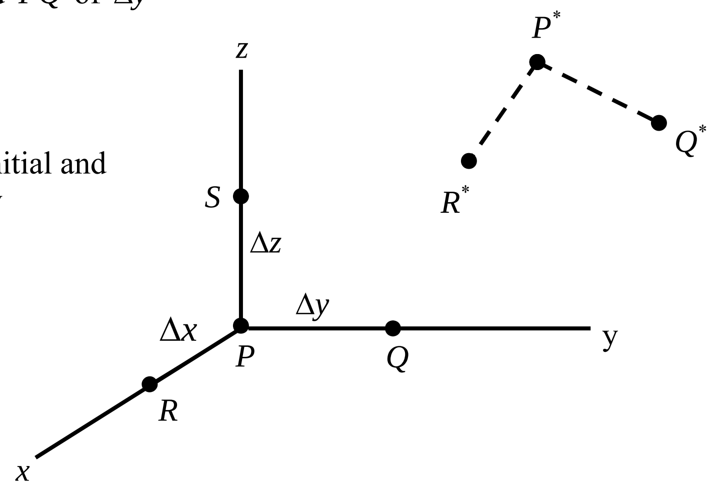
Where with coalescence of P & Q, we may drop subscript P :

$$\frac{(\overline{P^*Q^*})_y - \Delta y}{\Delta y} = \frac{\partial u_y}{\partial y} = \varepsilon_{yy}$$

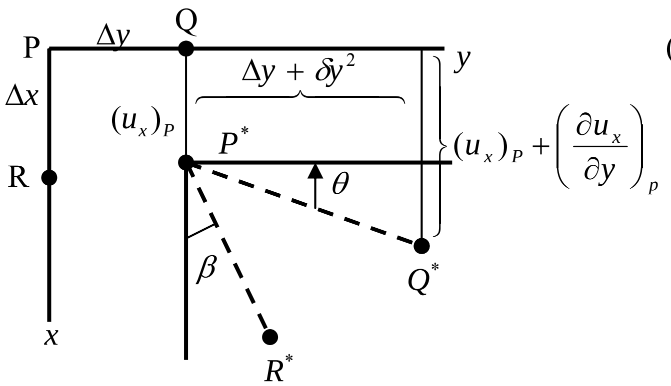
$\therefore$ ε_{yy} = change in length in the y direction per uint original length of vanishingly small line segment originally in the y direction.

Now consider $\overline{PR}$ of Δx and $\overline{PQ}$ of Δy

Line segments in initial and deformed geometry



We are interested in the projection of $\overline{P^*R^*}$ and $\overline{P^*Q^*}$ on to the xy place (on to the place the line segments were in undeformed state)



$(u_x)_P$ = displacement component of P in x direction

$(u_x)_P + \left(\frac{\partial u_x}{\partial y}\right)_p \Delta y + \ldots$ = displacement component of Q in the x direction

δy^2 = second order increment for a small deformation

$$tg\theta = \frac{\left(\frac{\partial u_x}{\partial y}\right)_p \Delta y + \ldots}{\Delta y + \delta y^2} \Rightarrow tg\theta = \theta = \frac{\partial u_x}{\partial y}$$

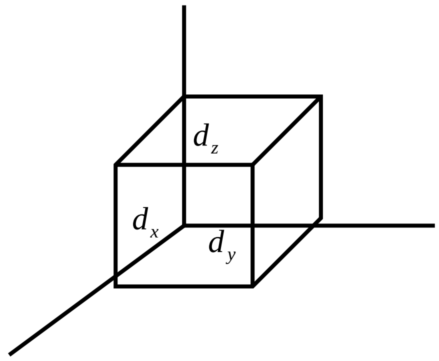
$$\Delta y \to 0$$

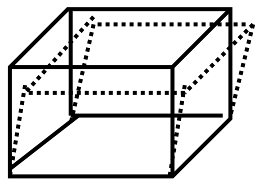
similarly : $\beta = \frac{\partial u_y}{\partial x}$

$$\theta + \beta = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 2\varepsilon_{xy} = \gamma_{xy}$$

γ_{ij} = change from a right angle of vanishingly small line segments originally in the $i \& j$ directions at a point

Now effect of strain on a infinitesimal rect. Parallelepiped in the undeformed geometry.





Zero shear stress means side will remain orthogonal on deformation. However position and orientation of the element may change as length of the sides and volume.

Existence of shear stress means sides may lose they mutual perpendicularity, (parallelograms instead of rectangles)
$\therefore$ Size of the rectangular parallelepiped is changed by normal strain while the basic shape is changed by shear strain.

$$\text{Prove : } \left(\frac{\Delta V}{V} = \varepsilon_{ii} \right)$$

2-2-2- The Rotation Tensor

Previously, we considered stretching of a line element to generate ε_{ij} and then used the deformation of a vanishingly small rectangular parallelepiped to give physical interpretation to the component of strain tensor.

We now introduce rotation tensor. This time rather than considering just the stretch of a vanishingly small line element, we consider the complete mutual relative motion of the end points of line element. (include rotation as well as stretching)

Consider $\overline{PN}$ the relative movement of end points can be given by using disp. field.

$$U_N - U_P = \left[U_P + \left(\frac{\partial u}{\partial x_j} \right)_p \Delta x_j + \ldots \right] - U_P$$

Expand U_N as a Taylor series about P

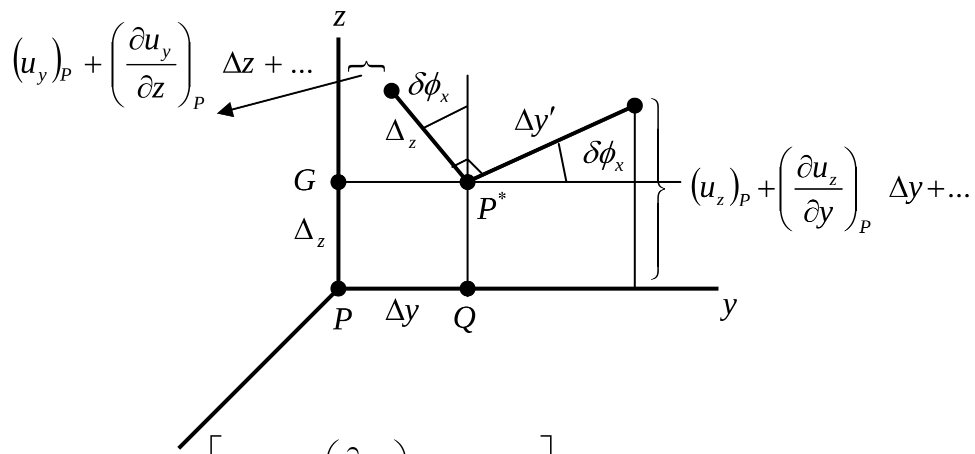
In limit $\Delta x_j \to 0 \quad du = \frac{\partial u}{\partial x_j} dx_j \quad \left(\text{index notation } du_i = \frac{\partial u_i}{\partial x_j} dx_j \right)$

Thus the relative movement du_i between the two adjacent point dx_i a part is

$$u_{i,j} = \underbrace{\frac{1}{2}(u_{i,j} + u_{j,i})}_{\varepsilon_{ij}} + \underbrace{\frac{1}{2}(u_{i,j} - u_{j,i})}_{w_{ij}}$$

ε_{ij} $\quad w_{ij}$ (rotation tensor skew-symetric)

Assume Rigid body motion $\overline{PQ}, \overline{PG}$; same $\delta\phi_x$



$$Sin\ \delta\phi_x = \frac{\left[(u_z)_P + \left(\frac{\partial u_z}{\partial y} \right)_P \Delta y + \ldots \right] - (u_z)_P}{\Delta y'}$$

$\Delta y \to 0 \quad (\Delta y = \Delta y')$

$$\delta\phi_x = \frac{\partial u_z}{\partial y}$$

$$PG : \to Sin\ \delta\phi_x = \frac{(u_y)_P - \left[(u_y)_P + \left(\frac{\partial u_y}{\partial z} \right)_P \Delta z + \ldots \right]}{\Delta z'}$$

$$\delta\phi_x = -\frac{\partial u_y}{\partial z}$$

$$\rightarrow \delta\phi_x = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) = w_{32} = -w_{23}$$

For other 2 components:

$$\delta\phi_y = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) = w_{13} = -w_{31}$$

$$\delta\phi_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = w_{21} = -w_{12}$$

For rigid body movement, the nonzero components of the rotation tensor give the infinitesimal rotation components of the element. What does w_{ij} represent when the rectangular parallelepiped is undergoing a movement including deformation of the element and not just $R.B$ rotation? Each line segment in the rectangular volume has its own angle of rotation and we can show that w_{ij} for such situation gives the average rotation components of all the line segments in the body. However we shall term the component of w_{ij} the rigid body rotation components.

From experiment ε_{ij} portion of equation w_{ij} related to the stress τ_{ij}

Further investigation: Transformation equation for strain.

2-2-3- Compatibility equations
Strain-displacement relations

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \quad (*)$$

If u_i's are know, ε_{ij} can be obtained.
The inverse problem of finding the displacement field from a strain field is not so simple.

Three functions u_i must be determined by integral of 6 partial differential equations (*) to ensure single-valued continuous solution u_i, we must impose certain restriction of ε_{ij}

⟹ can not set forth any ε_{ij}. to expect unique solution, the following equations are to be satisfied:

$$\begin{cases} \dfrac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \dfrac{\partial^2 \gamma_{xy}}{\partial x \partial y} \text{(2 more equations)} \\ 2\dfrac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \dfrac{\partial}{\partial x}\left(-\dfrac{\partial \gamma_{yz}}{\partial x} + \dfrac{\partial \gamma_{xz}}{\partial y} + \dfrac{\partial \gamma_{xy}}{\partial z} \right) \text{(2 more equations)} \end{cases}$$

total 6 equations

2.3 HOOKE'S Law

Linear elastic behavior

$\tau_{ij} = C_{ijkl}\ \varepsilon_{kl}$ generalized Hook law

$\tau_{ij}\ ,\ \varepsilon_{ij}\ \ 2^{nd}\ order\ tensor \Rightarrow C_{ijkl}\ \ 4^{th}\ order\ tensor$

$\tau_{ij}\ \ symmetric \Rightarrow C_{ijkl}\ \ symmetric\quad C_{ijkl} = C_{jikl}$

$\varepsilon_{kl}\ \ symmetric \Rightarrow C_{ijkl} = C_{ijlk}$

It can be shown that $C_{ijkl} = C_{klij}$ (Using Energy Concept It can be proved.) Thus, starting with 81 terms for C_{ijkl} ($=3^4$), we may show, using the three aforementioned symmetry relations for C_{ijkl} , that only 21 of these terms are independent.

We will assume now that the material is homogeneous (which has same composition throughout) so we may consider C_{ijkl} to be a set of constants for a given reference.

For an isotropic material, in which the material properties at a point are not dependent on direction, we have:

$$\tau_{ij} = \lambda\, \delta_{ij}\ \varepsilon_{ee} + 2G\varepsilon_{ij}$$

This is the general form of Hooke's law giving stress components in terms of strain components for isotropic materials. The constant λ and G are the so-called Lame constants. It can be seen that as a result of isotropy the number of independent elastic moduli has been reduced fron 21 to 2. The inverse of Hooke's law yielding:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\ \tau_{ij} - \frac{\nu}{E}\ T_{kk}\ \delta_{ij}$$

E and ν are Young's modulus and the poisson ratio stemming from one-dimensional test data.

$$\varepsilon_{xx} = \frac{1}{E}\left[\tau_{xx} - \nu\left(\tau_{yy} + \tau_{zz}\right)\right] \qquad \varepsilon_{xy} = \frac{1+\nu}{E}\,\tau_{xy} = \frac{1}{2G}\tau_{xy}$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\tau_{yy} - \nu\left(\tau_{xx} + \tau_{zz}\right)\right]$$

$$G = \frac{E}{2(1+V)} \qquad \lambda = \frac{EV}{(1+V)(1-2V)} \qquad \nu = \frac{\lambda}{2(\lambda+G)} \qquad E = \frac{G(3\lambda+2G)}{\lambda+G}$$

3- Boundary-value problems for linear elasticity

The complete system of equations for linear elasticity for homogeneous, isotropic solid includes the equilibrium equations:

$$\tau_{ij,j} + B_i = 0 \qquad (3\ equations\ )$$

The stress-strain law:

$$\tau_{ij} = \lambda\,\varepsilon_{ll}\delta_{ij} + 2G\varepsilon_{ij} \quad \oplus \qquad (6\ equations)$$

Strain displacement relations:

$$\varepsilon_{ij} = \frac{1}{2}\,(u_{i,j} + v_{j,i}) \qquad * \quad (6\ equations)$$

We have 15 equations and 15 unknowns. When explicit use of the displacement field is not made, we must be sure that the compatibility equations are satisfied.

It must be understood that B_i and $T_i^{(v)}$ have resultants that satisfy equilibrium equations for the body as dictated by Rigid body mechanics. In this regard that B_i and $T_i^{(v)}$ must be statically compatible.

We may pose three classes of boundary values problems:

1st kind B.V. problem: determine the distribution of stresses and displacements in the interior of the body under a given body force distribution and a given surface traction over the boundary.

2nd kind B.V. problem: determine the distribution of stresses and displacements in the interior of the body under the action of a given body force distribution and a prescribed displacement distribution over the entire boundary.

Mixed B.V. problem: determine the distribution of stresses and displacements in the interior of the body under the action of a given body force distribution with a given traction distribution over part of the boundary (s_1) and a prescribed displacement distribution over the remaining part of the boundary s_2.

Note : on the surfaces where the $T_i^{(\nu)}$ are prescribed, Cauchy's formula $T_i^{(\nu)} = T_{ij}\,\nu_j$ must apply.

1st kind: convenient to express basic equations in terms of stresses. To do this:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\,\tau_{ij} - \frac{\nu}{E}\,\tau_{kk}\,\delta_{ij} \xrightarrow{\text{substitutide}} \text{in compatibility euations}$$

Using equilibrium equations, we can arrive at the Beltrami-Michell system of equations:

$$\nabla^2\,\tau_{ij} + \frac{1}{1+\nu}\,K_{,ij} - \frac{\nu}{1+\nu}\,\delta_{ij}\quad \nabla^2 K = -(B_{i,j} + B_{j,i})$$

where $K = \tau_{kk}$

The solution of these equations, subject to the satisfaction of Cauch's formula on the boundary for simply connected domains, will lead to a set of stress components that both satisfy the equilibrium equations and are derivable from a single-valued continuous displacement field.

2nd kind: Substitute equations * and ☺ in the equilibrium equations to yield differential equations with the displacement field as the dependent variable. Then we get Navier equations of elasticity:

$$G\nabla^2 u_i + (\lambda + G)u_{j,ji} + B_i = 0 \ \checkmark$$

For dynamic conditions we need only employ the following equations in place of the equilibrium equations.

$$\tau_{ij,j} + B_i = \rho\, \dot{u}_i$$

The results are the addition of the term $p\, \ddot{u}_t$ on the right side of the above equations. If the above equation can be solved in conjunction with the prescribed displacements on the surface and if the resulting solution is singled –valued and continuous the problem may be considered solved.

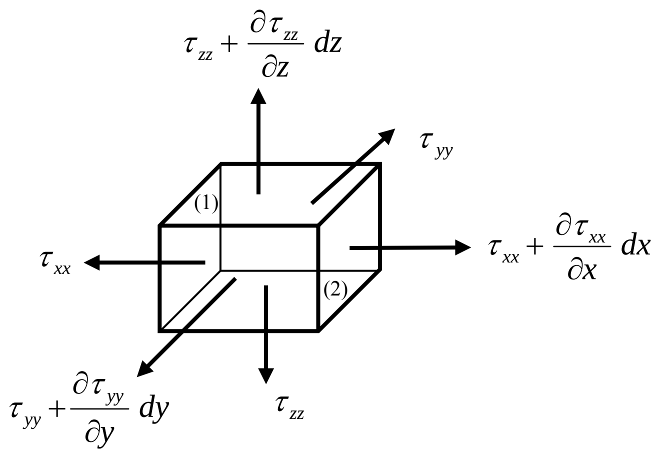
Solution for mixed BV problems will be investigated using different techniques introduced partly in this notes such as variational approach.

4- Energy consideration

We have described the stress tensor arising from equilibrium consideration and the strain tensor from kinematics considerations. These tensors are related to each other by laws that are called constitutive laws. In general such relations include temperature and time as other variables. In addition, they often require knowledge of the history of deformation lending to the instantaneous condition of interest in order to properly relate stress and strain. We assume that the constitutive laws relate stress and strain directly and uniquely. That is,

$$\tau_{ij} = \tau_{ij}(\varepsilon_{11}, \varepsilon_{12}, ..., \varepsilon_{33}) \qquad \textit{Constitutive law (C.L.)}$$

Consider an infinitesimal rectangular element under the action of normal stresses only.



The displacements of faces 1 and 2 in the x direction are u_x as $u_x + \frac{\partial u_x}{\partial x} dx$,

Increment of mechanical work done by the stresses on the element during deformations is:

$$-\tau_{xx}\, du_x\, dy\, dz + \left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial_x} dx\right) d\left(u_x + \frac{\partial u_x}{\partial x} dx\right) dy\, dz \; +$$

$$B_x\, d_x\, d_y\, d_z \quad d\left(u_x + k\frac{\partial u_x}{\partial x} d_x\right) \qquad 0 \langle k \langle 1$$

Canceling terms and deleting the higher order expressions:

$$\left[\tau_{xx}\, d\left(\frac{\partial u_x}{\partial x}\right) + \underbrace{\left(\frac{\partial T_{xx}}{\partial x} + B_x\right)}_{equilibrium\, =\, 0} du_x\right] d_x\, d_y\, d_z$$

$$\tau_{xx}\ d\left(\frac{\partial u_x}{\partial x}\right) d_x\, d_y\, d_z = \tau_{xx} d\varepsilon_{xx}\ dV$$

Similar expression for y and z directions can be obtained. Thus for normal stresses on an element, the incremental of mechanical work for isotropic materials is:

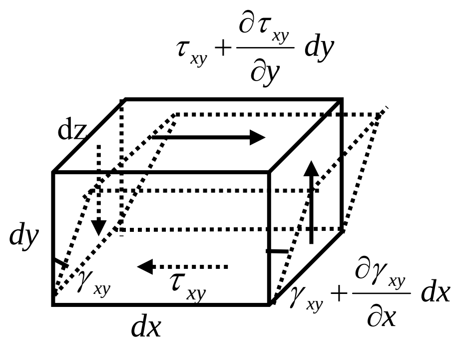
$$\left(\tau_{xx} d\varepsilon_{xx} + \tau_{yy} d\varepsilon_{yy} + \tau_{zz} d\varepsilon_{zz}\right) dV \qquad (normal\ \ stresses)$$

$w =$ Mechanical work per unit volume

$$dw = \tau_{xx} d\varepsilon_{xx} + \tau_{yy} d\varepsilon_{yy} + \tau_{zz} d\varepsilon_{zz}$$

Now consider the case of pure shear:

The mechanical increment of work



$$\left[\left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy\right) dzdx\right] \left\{ d\left(\gamma_{xy} + \beta \frac{\partial \gamma_{xy}}{\partial x} dx\right) dy \right\}$$

$$+ B_x \quad dx\,dydz \quad d\left(\gamma_{xy} + \eta \frac{\partial \gamma_{xy}}{\partial x} dx\right)\ \left(k\ dy\right) \qquad 0 \le \beta, \eta, k \le 1$$

$$\rightarrow \tau_{xy} d\gamma_{xy} dxdydz = 2\ \tau_{xy} d\varepsilon_{xy} dV$$

Thus, for pure shear stresses on all faces we get the following result for increments of mechanical work:

$$2\ \left(\tau_{xy} d\varepsilon_{xy} + \tau_{xz} d\varepsilon_{xz} + \tau_{yz} d\varepsilon_{yz}\right) dV$$

Mechanical work increment per unit volume at a point for a general state of stress is:

$$dw = \tau_{ij} d\varepsilon_{ij} \qquad \text{(valid only for infinitesimal deformation)}$$

Now integrating from 0 to some strain level ε_{ij} we get:

$W = \int_0^{\varepsilon_{ij}} \tau_{ij} d\varepsilon_{ij} = u =$ strain energy density function which is the mechanical work performed on an element per unit volume at a point during a deformation.

$$du = \tau_{ij} d\varepsilon_{ij} \Rightarrow \frac{\partial u}{\partial \varepsilon_{ij}} = \tau_{ij}$$

(u is point function, integral independent of path then perfect differential)

Total strain energy $U = \iiint_v \left[\int_0^{\varepsilon_{ij}} \tau_{ij} d\varepsilon_{ij} \right] dV$

$$w = \int_0^{\varepsilon_{ij}} \tau_{ij} d\varepsilon_{ij} = \int \tau_{xx} d\varepsilon_{xx} + \tau_{yy} d\varepsilon_{yy} + \tau_{zz} d\varepsilon_{zz} + 2(\tau_{xy} d\varepsilon_{xy} + \tau_{xz} d\varepsilon_{xz} + \tau_{yz} d\varepsilon_{yz})$$

$$U = \iiint_v w \; dV$$

Examples of Calculating Total strain energy

Uniaxial stress

$\sigma = E\varepsilon$ $\quad d\varepsilon = \frac{d\sigma}{E}$

$$U = \iiint_v \left(\int_0^{\varepsilon_{ij}} \sigma_{xx} d\varepsilon_{xx} \right) dV$$

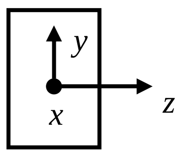
$$w = \int \sigma_x d\varepsilon_x = \int \frac{\sigma_x}{E} d\sigma_x = \frac{1}{2} \frac{\sigma_x^{\;2}}{E} \qquad w = \frac{1}{2} \frac{\sigma_x^{\;2}}{E}$$

$$U = \iiint \frac{1}{2} \frac{\sigma_x^{\;2}}{E} dV$$

Pure bending

$\sigma = \frac{MY}{I}$

$\sigma = E\varepsilon$



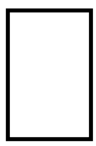
$$w = \int_0^{\varepsilon_{ij}} \sigma_{xx} d\varepsilon_{xx} = \frac{1}{2} \frac{\sigma_x^{\;2}}{E} dy$$

$$U = \iiint \frac{1}{2E} \left(\frac{MY}{I} \right)^2 dV$$

$$= \int \frac{1}{2E} \frac{M^2}{I^2} dx \int \int Y^2 dA = \frac{1}{2E} \int \frac{M^2}{I} dx$$

$$M = \frac{EI}{\rho} = EIW_{xx}^{\;2} \qquad U = \frac{1}{2} \int_o^l E \; I \; W_{xx}^{\;2} dx$$

Shear stress



$$W = \int \tau_{ij} d\varepsilon_{ij} = \int \tau_{ij} \frac{d\tau_{ij}}{G} = \frac{1}{2} \frac{\tau_{ij}^{\;2}}{G}$$

$\tau_{xy} = G\gamma_{xy}$

$$U = \frac{1}{2} \int \tau_{ij}^{\;2} / G \; dV$$

5- Principles of virtual work

Particle Mechanics : Virtual work is defined as the work done on a particle by all the forces acting on the particle as this particle is given a small hypothetical displacement , a virtual displacement , which is consistent with the constraints present.
The applied forces are kept constant during the virtual displacement.

Deformable body: Same as particle with specifying a continuous displacement field with small deformation and constraint, applied force kept constant. We conveniently denote a virtual displacement by employing the variational operation δ .

In general situation we would have as load possibilities a body force distribution B_i through out the body as well as surface tractions $T_i^{(v)}$ over part of the boundary , S_1, of the body. Over the remaining part of the boundary, S_2, we have prescribed the displacement field u_i ,in which case, to avoid violating the constraints we must be sure that $\delta u_i = 0$ on S_2.
Virtual work for such a general solution would be:

$$\delta W_{virt} = \iiint_v B_i\, \delta u_i\, dv + \oint\!\!\int_s T_i^{(v)}\, \delta u_i\, ds$$

B_i and $T_i^{(v)}$ must not depend on δu_i in computation of δW_{virt}. We can expand the surface integral to cover entire surface since $\delta u_i = 0 \;\; on \;\; S_2$, thus $S = S_1 + S_2$
We now develop the principle of virtual work for a deformable body

$$\delta W_{virt} = \iiint_v B_i\, \delta u_i\, dv + \oint\!\!\int_s \tau_{ij}\, \upsilon_j\, \delta u_i\, ds$$

$$= \iiint_v B_i\, \delta u_i\, dV + \iiint_v (\tau_{ij}\, \delta u_i)_{,j}\, dV$$

$$= \iiint_v (B_i + \tau_{ij,j})\, \delta u_i\, dV + \iiint_v \tau_{ij}\, (\delta u_i)_{,j}\, dV$$

We now introduce a kinematically compatible strain field variation $\delta\varepsilon_{ij}$ (it is because it is formed directly from the displacement field variation).

$$(\delta u_i)_{,j} = \delta(u_{i,j}) = \delta(\varepsilon_{ij} + W_{ij}) = \delta\varepsilon_{ij} + \delta W_{ij}$$

Because of skew symmetry of the rotation tensor and the symmetry of the stress tensor, $\tau_{ij}\,\delta w_{ij} = 0$

$$\tau_{ij}\,(\delta u_i)_{,j} = \tau_{ij}\,\delta\varepsilon_{ij}$$

$$\Rightarrow \iiint_v B_i\,\delta u_i\,dV + \oint\!\!\int_s T_i^{(v)}\,\delta u_i\,ds = \iiint_v (\tau_{ij,j} + B_i)\,\delta u_i\,dV + \iiint_v \tau_{ij}\,\delta\varepsilon_{ij}\,dV$$

We now impose the condition that we have static equilibrium. This means in the above equation that:

1. External load B_i and $T_i^{(v)}$ are such that there is overall equilibrium for the body from the point of view rigid body mechanics we say that B_i and $T_i^{(v)}$ are statically compatible.
2. At any point in the body $T_{ij,j} + B_i = 0$

$$\Rightarrow \underbrace{\iiint_v B_i\,\delta u_i\,dV + \oint\!\!\int_s T_i^{(v)}\,\delta u_i\,ds}_{\text{external virtual work}} = \underbrace{\iiint_v T_{ij}\,\delta\varepsilon_{ij}\,dV}_{\text{internal virtual work}}$$

This is the principle of v.w. for a deformable body

We can say that necessary condition for equilibrium is that for any kinematically compatible deformation field $(\delta u_i, \delta\varepsilon_{ij})$, the external v.w. with statically compatible body forces and surface traction , must equal the internal v.w.

This is sufficient for equilibrium.

Another more useful interpretation of the principle of v.w. is as follows.

The necessary requirements for equilibrium of a particular stress field τ_{ij} are that :

1. B_i and $T_i^{(v)}$ are statically compatible
2. The particular stress field τ_{ij} satisfies the v.w. equilibrium for any kinematically compatible, admissible, deformation field.

Note: the mathematical relation between a deformation field and a stress field is independent of any constitutive law and applies to all materials within the limitations of small deformation.

We have shown that the satisfaction of the principle of v.w. is a necessary relation between the external loads and stresses in a body in equilibrium.

We can also show that satisfaction of the principle of v.w. is <u>sufficient</u> to satisfy the equilibrium requirement of a body.

Assume v.w. equilibrium is valid

$$\iiint_v \tau_{ij}\,\delta\varepsilon_{ij}\,dV = \iiint_v \tau_{ij}\,\delta\left(\frac{u_{i,j}+u_{j,i}}{2}\right)dV = \iiint_v \tau_{ij}\frac{(\delta u_i)_{,j}}{2}\,dV + \iiint_v \tau_{ij}\frac{(\delta u_j)_{,i}}{2}\,dV$$

$$= \iiint_v \tau_{ij}\,(\delta u_i)_{,j}\,dV$$

We made use of symmetry of τ_{ij}. We can write the last expression as follows:

$$\iiint_v \tau_{ij}\,(\delta u_i)_{,j}\,dV = \iiint_v (\tau_{ij}\,\delta u_i)_{,j}\,dV - \iiint_v \tau_{ij,j}\,\delta u_i\,dV$$

Using divergence theorem

$$\iiint_v \tau_{ij}\,(\delta u_i)_{,j}\,dV = \iint_s \tau_{ij}\,\delta u_i\,\upsilon_j\,ds - \iiint_v \tau_{ij,j}\,\delta u_i\,dV$$

$$= \iint_{s_1} \tau_{ij}\,\delta u_i\,\upsilon_j\,ds - \iiint_v \tau_{ij,j}\,\delta u_i\,dV$$

We have made use of the fact that $\delta u_i = 0 \quad on \quad S_2$

Now substituting these results for the last integral in the principle of virtual work, that was found previously and is as the following:

$$\iiint_v B_i\,\delta u_i\,dV + \oint\!\!\int_s T_i^{(v)}\,\delta u_i\,ds = \iiint_v \tau_{ij}\,\delta\varepsilon_{ij}\,dV$$

Results in the followings:

$$\iiint_V (\tau_{ij,j}+B_i)\,\delta u_i\,dV + \oint\!\!\int_s (T_i^{(v)} - \tau_{ij}\upsilon_j)\,\delta u_i\,ds = 0$$

Since δu_i is arbitrary, we must conclude $\tau_{ij,i} + B_i = 0 \quad in \quad V$

By the same reasoning $T_i^{(v)} = \tau_{ij}\,\upsilon_j \quad on \quad S_1$

We have generated Newton's law for equilibrium at any point inside the body and Cauchy's formula, which ensure equilibrium at the boundary.

$\Rightarrow$ Satisfaction of principle of v.w. is both necessary and sufficient for equilibrium.

6- The Method of Total Potential Energy

Note: Calculus of Variations has to be reviewed.

We now develop from the virtual work idea, the concept of total potential energy which applies to elastic body (not necessary linear elastic):

$$\iiint_v B_i\, \delta u_i\, dV + \oint\!\!\int_s T_i^{(v)}\, \delta u_i\, ds = \iiint_v \tau_{ij}\, \delta\varepsilon_{ij}\, dV$$

$$du = \tau_{ij}\, d\varepsilon_{ij} \Rightarrow \frac{\partial u}{\partial \varepsilon_{ij}} = \tau_{ij} \Rightarrow \frac{\partial u}{\partial \varepsilon_{ij}} \delta\varepsilon_{ij} = \delta^1 u$$

$$\iiint_v B_i\, \delta u_i\, dV + \oint\!\!\int_s T_i^{(v)}\, \delta u_i\, ds = \iiint_v \delta^1 u\, dV = \delta^1 \iiint_v u dV = \delta^1 U$$

Note: δu_i is virtual displacement field. A priori not related to stress field

We define potential energy V of applied load as a functional of displacement field u_i

$$V = -\iiint_v B_i\, u_i\, dV - \oint\!\!\int_s T_i^{(v)}\, u_i\, ds \quad B_i \text{ and } T_i^{\nu} \text{ prescribed}$$

$$\delta^1 V = -\iiint_v B_i \frac{\partial u_i}{\partial u_j} \delta u_j\, dV - \oint\!\!\int_s T_i^{(v)} \frac{\partial u_i}{\partial u_j} \delta u_j\, ds$$

$$\frac{\partial u_i}{\partial u_j} = \delta_{ij}$$

$$\delta^1 V = -\iiint_v B_i\, \delta u_i\, dV - \oint\!\!\int_s T_i^{(v)}\, \delta u_i\, ds$$

$\delta^1 (U + V) = 0$ (Principle of total potential energy)

$\pi = U + V$ (Total Potential Energy)

$$\pi = U - \iiint_v B_i\, u_i\, dV - \oint\!\!\int_s T_i^{(v)}\, u_i\, ds$$

$\delta^1 (\pi) = 0$ Principle of total potential energy

Interpretation: The necessary requirements for equilibrium of a particular stress field τ_{ij} :

1. B_i and T_i^{ν} are statically compatible

2. The deformation field, to which the field τ_{ij} is related through a constitutive law for elastic behavior, extremize TPE with respect to all other kinematically compatible admissible deformation fields.

Extremization of the TPE w.r.t admissible deformation fields is necessary for equilibrium to exist between the forces and the stresses in a body. Just in the method of virtual work, we can show it to be a sufficient condition for equilibrium.

We can show that TPE is actually a local minimum for the equilibrium configuration under loads B_i and T_i^{ν} compared with the TPE corresponding to neighboring admissible configurations with the same B_i and T_i^{ν}.

Examine the difference between TPE of equilibrium state and an admissible neighboring state $u_i + \delta u_i$ and $\varepsilon_{ij} + \delta\varepsilon_{ij}$ show that the second variation of TPE is positive.

The total potential energy theorem states that' of all the admissible fields which satisfy compatibility and essential boundary conditions, the actual one which satisfies equilibrium and stress BC's provide a minimum to π.

The total potential (π) is also called the functional of the problem.

Assume that in the functional (π) the highest derivative of a state variable (wrt a space coordinates) is of order m, i.e. the operator contains at most m^{th} order derivatives. Such a problem we call C^{m-1}variational problem.

Considering the boundary of the problem, we can identify two classes on bc's:

Essential bc's (geometric): correspond to prescribed displacement and rotations. The order of the derivatives in the essential bc's is in a C^{m-1} Problem, at most m-1.

Natural boundary conditions (force bc's): corresponds to prescribed boundary force and momentums. The highest derivative in this bc's are of order m to 2m-1.

By invoking the stationary of the functional a problem, the problem governing differential equation and natural and essential bc's can be derived.

In C^{m-1} variational problem, the order of the highest derivative presented in the problem governing differential equation is 2m.

Therefore, integration by parts is employed m times.

Effect of bc's are included implicitly in π.

7- Differential Equations VS functional for continuous systems

We can get a solution to a partial differential equation which is satisfied at each point in the body and also satisfy a set of boundary conditions. A solution obtained, maybe for displacements or stresses, etc.

A functional represents a number (scalar) and for naturally occurring functional, it may represent work, energy or power or etc. In some instances, it may not represent any physical quantity. At extremum, it yields a solution to the differential equation (equilibrium or momentum balance or heat balance, etc.).

$$e.g. \qquad I = \int f(y)\,dx \qquad (functional)$$

Existence of a functional and solution obtained as extremum of this functional also helps to determine as to what kind of equilibrium is achieved. This leads to theory of stability, for example, if it is a minimum at extremum then the solution obtained is stable!

To go from differential equation to variational problem we need to know operational algebra or calculus (functional analysis) and to go from variational problem to differential equation we need to know the calculus of variations.

7.1 Formulation of continuous systems

We consider a typical differential element with the objective of obtaining differential equations that express the element equilibrium requirements, constitutive relations, and element interconnectivity requirements. These differential equations must hold throughout the domain of the system and before the solution can be obtained they must be supplemented by boundary conditions and, in dynamic analysis, also by initial conditions.

Two different approaches can be followed to generate the system governing differential equations.

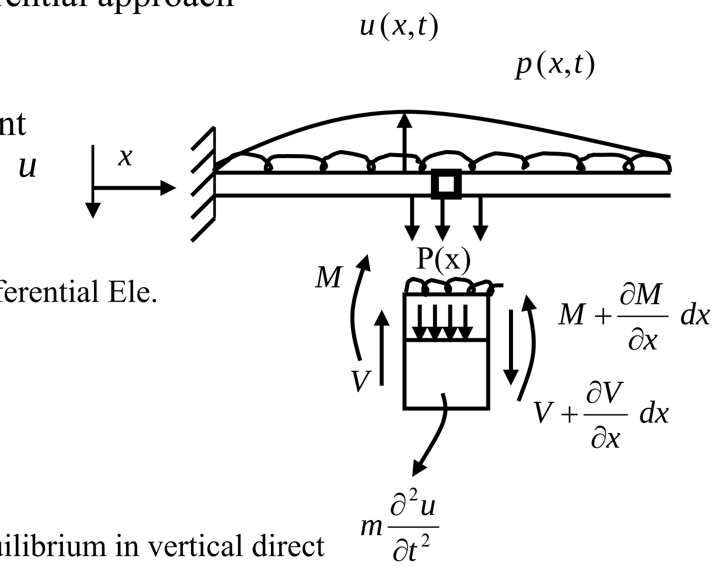
1. The direct method (differential equations)
2. The variational method

The direct method

In this method, we establish the equilibrium and constitutive requirements of typical differential elements in terms of state variables. These considerations lead to a system of differential equations in the state variables. In general the equations must be supplemented by additional differential equations that impose appropriate constraints on the state variables in order that all compatibility requiremvents be satisfied. Finally to complete the formulation all the boundary conditions and in a dynamic analysis the initial conditions are stated in differential formulation for a continous system, a differential element with objective of obtaining differential equation that express element equilibrium is found. This differential equation must hold through the domain of the system. The D.E must be supplemented by B.C.'S and dynamic analysis, initial condition example.

7.1.1 Examples of differential approach

Example 1- Beam element



a) Differential Ele.

b) Equilibrium in vertical direct

$$\frac{\partial V}{\partial x} dx + p dx = m \frac{\partial^2 u}{\partial t^2} dx$$

or

$$\boxed{\frac{\partial V}{\partial x} - m \frac{\partial^2 u}{\partial t^2} + p = 0}$$

equating sum of the moment about the left hand face to zero

$$\left(V+\frac{\partial V}{\partial x}\,dx\right)dx+pdx\,\frac{dx}{2}+m\frac{\partial^2 u}{\partial t^2}\,dx\,\frac{dx}{2}-M-\frac{\partial M}{\partial x}\,dx+M=0$$

$$V+\frac{\partial M}{\partial x}=0 \qquad *$$

now $\theta=\frac{\partial u}{\partial x}$ also from elementary beam theory $M=EI\frac{\partial\theta}{\partial x}$

$$\theta\Rightarrow\; M=EI\frac{\partial^2 u}{\partial x^2} \qquad \Rightarrow^{*}\; V=-\frac{\partial}{\partial x}\left(EI\frac{\partial^2 u}{\partial x^2}\right)$$

$$\Rightarrow\; \boxed{\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right)+m\frac{\partial^2 u}{\partial t^2}=P}$$

Transverse vibrate of beam

For a unique solution we must specify bc's

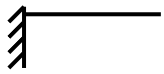
$u=0$ @ $x=0$

$u=0$ @ $x=l$

$M=EI\frac{\partial^2 u}{\partial x^2}=0$  Simply supported.

$M=0$

$V=0$  free

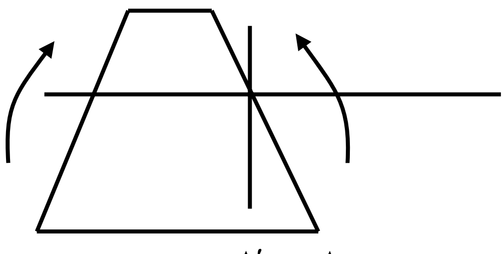


$u=0$

$\theta=\frac{\partial u}{\partial x}=0$

Note of the elementary beam theory
Plain remain plain

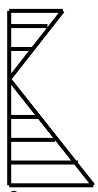
$$\boxed{M=EI\frac{d\theta}{\partial x}}$$



$\Sigma F=0$

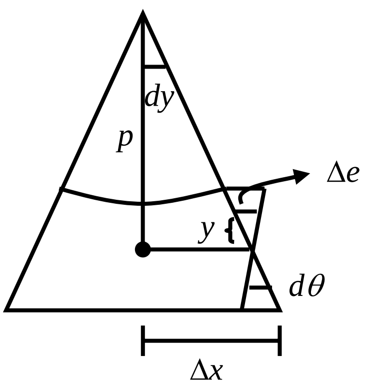
$\Rightarrow \int ydA=0$

$\int -\frac{y}{c}\sigma_{\max}(-dA)=0$



$M=\int\left(-\frac{y}{c}\sigma_{\max}\right)dAy$

$\sigma=\frac{My}{I}$

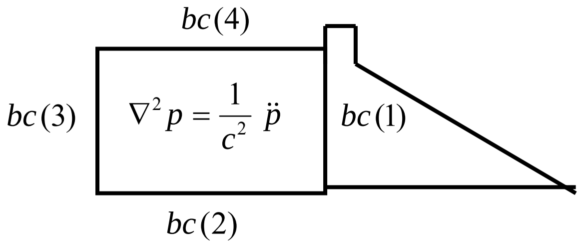


$\frac{My}{I}=E\varepsilon$ (Hook's

$\frac{M}{EI}=\frac{\varepsilon}{y}=\frac{(\Delta e/\Delta x)}{y}=\frac{\Delta e/y}{\Delta x}=\frac{d\theta}{dx}=\frac{1}{\rho}$

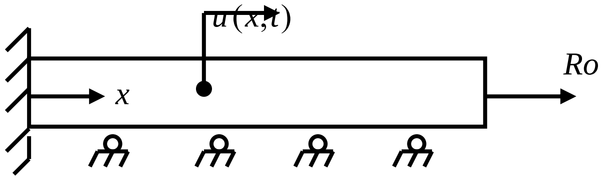
$\Rightarrow$ $1/\rho = \frac{M}{}$ ... $\frac{1}{\rho}=\frac{d^2x}{}$

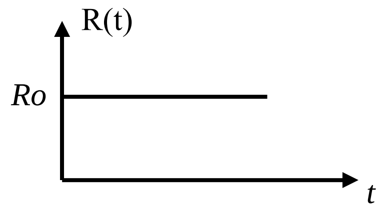
Example 2- Dam's Reservoir



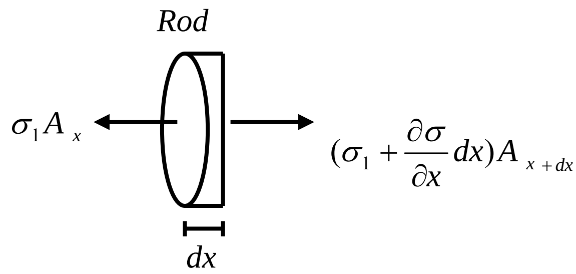
Example 3- Rod subjected to step load

E young modulus p
ρ mass density
A cross section





1) Differential element



Equilibrium $$\left(\sigma + \frac{\partial \sigma}{\partial x}\,dx\right)A - (\sigma A) = A\rho dx \quad \ddot{u}$$

Constitutive relation $$\sigma = E\,\frac{\partial u}{\partial x}$$

Combining equations: $$\frac{\partial^2 u}{\partial x^2} = \frac{1}{C^2}\,\frac{\partial^2 u}{\partial t^2} \qquad C = \sqrt{\frac{E}{\rho}}$$

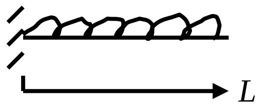
b.c's $u\,(o,t) = 0 \qquad EA\,\frac{\partial u}{\partial x}\,(l,t) = Ro$

Initial load $u(x,o) = 0 \qquad \frac{\partial u}{\partial t}\,(x,o) = 0$

7-1-2 Examples of Variational approach

Example1. Beam

$$\Pi(w) = \underbrace{\frac{EI}{2}\int_0^L w_{xx}^2\,dx}_{U(w)\ \text{strain energy}} - \underbrace{\int_0^L p(x)\,w(x)\quad dx}_{\text{potential energy of loading}}$$



$$m = 2$$
$$C^{m-1} = C^1$$
$$\text{essential bc} \Rightarrow w, w_x$$

$$\delta\Pi = \frac{EI}{2}\int_0^L 2\,w_{xx}\,\delta w_{xx}\,dx - \int_0^L P\,\delta w \quad dx$$

(1) $\Rightarrow\ = EIw_{xx}\ \overset{(1)}{\delta w_x}\Big|_0^L - EI\int_0^L w_{xxx}\delta w_x \quad dx$

$$= EIw_{xx}\,\delta w_x\Big|_0^L - EIw_{xxx}\,\delta w\Big|_0^L + EI\int_0^L w_{xxxx}\,\delta w \quad dx$$

$$:\ \delta\Pi = \int_0^L (EIw_{xxxx} - P)\,\delta w\ dx + EIw_{xx}\ \delta w_x\Big|_0^L - EIw_{xxx}\,\delta w\Big|_0^L \Rightarrow$$

$$0 \le x \le l \qquad EIw_{xxxx} - P = 0$$

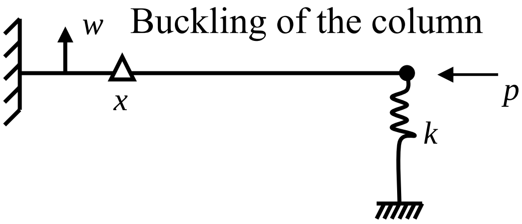
bc's $\quad EIw_{xx}\ \delta w_x\Big|_0^L = 0$

$$EIW_{xxx}\ \delta w\Big|_0^L = 0$$

In general on the bc's

at $\quad x=0 \quad or \quad x=L$

Example2:

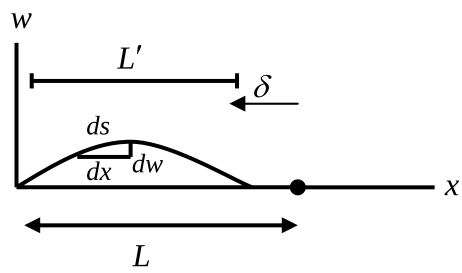


$$\Pi = \frac{1}{2}\int_0^L EI\,(w_{xx})^2\ dx - \underbrace{\frac{P}{2}\int_0^L w_x^2\ dx}_{\text{prove it *}} + \frac{1}{2}\,k\ w_L^2$$

$m = 2$

$C^{m-1} = C^1$

$ess.\ b.c \Rightarrow w, w_x$



* $U = \frac{1}{2}\int_0^L EI\ w_{xx}^2\ dx$

$w = P\,\delta \qquad \delta = L - L'$

$L = \int ds$ (*no change in length due to w*)

$$ds = \sqrt{dw^2 + dx^2} = \sqrt{1+\left(\frac{dw}{dx}\right)^2}\ dx \ \Rightarrow L = \int_0^{L'} \sqrt{1+w'^2}\ dx \ \ (0 \to L')$$

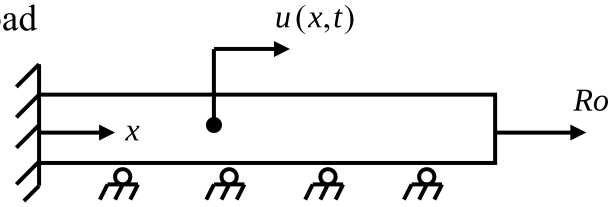
$$L \approx \int_0^{L'} \left(1 + \frac{1}{2}w'^2\right) dx \qquad \textit{for small disp.}$$

$$L \approx \int_0^{L'} dx + \frac{1}{2}\int_0^{L'} w'^2\ dx = L' + \frac{1}{2}\int_0^{L'} w'^2\ dx$$

$$\delta = L - L' = \frac{1}{2}\int_0^{L'} w'^2\ dx \qquad \delta \textit{ is small } L \approx L'$$

$$\boxed{W = \frac{P}{2}\int_0^L w'^2\ dx}$$

Example3. Rod subjected to STEP load



$$\Pi = \int_0^L \frac{1}{2} EA\,u_x^2\,dx - \int_0^L u\,f^B\,dx - u_L R$$

$bc's \quad u_0 = 0 = u(0,t)$

$u_L = u(L,t)$

$f^B = bod\ \ force\ \ for\ \ unit\ \ length$

$$\delta\Pi = 0 = \int EAu_x\,\delta u_x\,dx - \int_0^L \delta u\,f^B\,dx - \delta u_L R = 0$$

$$= EAu_x\,\delta u\Big|_0^L - \int_0^L EAu_{xx}\,\delta u\,dx - \int_0^L \delta u\ f^B\,dx - \delta u_L R$$

$$= -\int_0^L \left(EAu_{xx} + f^B\right)\delta u\,dx + \left[EAu_x\big|_L - R\right]\delta u_L - EAu_x\big|_{x=0}\ \delta u_0 = 0$$

δu *is arbitrary*

$\Rightarrow EA\,u_{xx} + f^B = 0$

$x = L \qquad EAu_x = R \qquad or \qquad \delta u_L = 0$

$x = 0 \qquad EAu_x = 0 \qquad or \qquad \delta u_0 = 0$

$\qquad\qquad x = 0$

$m = 1 \qquad C^0$ variational problem

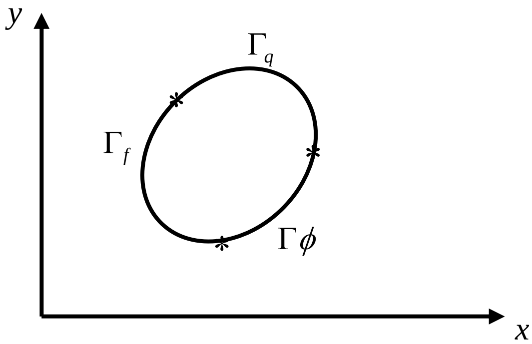
Example 4. 2-D Variational Principle

$$J = \int_{\Omega} \left[\frac{k}{2}\,\phi_x^2 + \frac{k}{2}\phi_y^2 - Q\phi \right] d\,\Omega - \int_{\Gamma_q} \bar{q}\;\phi\; d\,\Gamma$$

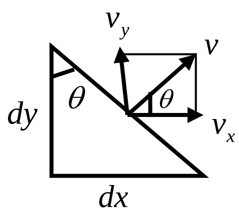
$K\ \&\ Q$ *are functions of positions only*

$\delta\phi = 0 \quad on \quad \Gamma\phi \quad (part\ of\ the\ boundary)$

$\bar{q}$ *specified on* Γ_q



$$\delta J = \int_{\Omega} \left[k\,\phi_x\;\delta\phi_x + k\,\phi_y\;\delta\phi_y - Q\,\delta\,\phi \right] d\,\Omega - \int_{\Gamma_q} \bar{q}\;\delta\phi\; d\,\Gamma$$



Note: $\delta\,\phi_x = \delta\,\dfrac{\partial\phi}{\partial x} = \dfrac{\partial}{\partial x}\,(\delta\phi) = (\delta\phi)_x$

Integrate by part the first two terms

$dy = d\,\Gamma\,\cos\,\theta$

$dx = d\,\Gamma\;\sin\,\theta$

$$\int_{\Omega} k\,\phi_x\;\delta\phi_x\;dxdy = \int_{\Omega} k\,\phi_x\,(\delta\phi)_x\;dxdy = \int_{\Gamma} k\,\phi_x\;\delta\phi\,\overset{v_x\,d\,\Gamma}{\overbrace{dy}} - \int_{\Omega} (k\phi_x)_x\;\delta\phi\;d\,\Omega$$

$$\int_{\Omega} k\,\phi_y\;\delta\phi_y\;dxdy = -\int_{\Omega} (k\,\phi_y)_y\;\delta\phi\;d\,\Omega + \int_{\Gamma} k\;\phi_y\;\delta\phi\;\underset{v_y\,d\,\Gamma}{\underbrace{dx}}$$

$$\delta\,J = \int_{\Omega} -\left[(k\,\phi_x)_x + (k\,\phi_y)_y + Q \right] \delta\phi\;d\,\Omega + \int_{\Gamma} \underset{k\,\frac{\partial\phi}{\partial\,n}}{\underbrace{\left[k\,\phi_x\,v_x + k\,\phi_y\,v_y \right]}}\,\delta\phi\;\;d\,\Gamma - \int_{\Gamma_q} \bar{q}\;\delta\phi\;\;d\,\Gamma$$

$$\oint_{\Gamma} k\phi_n \,\delta\phi \, d\Gamma = \int_{\Gamma_q} + \int_{\Gamma_\phi} + \int_{\Gamma_f} \qquad \begin{array}{l} \Gamma = \Gamma_f + \Gamma_q + \Gamma\phi \\ \delta\phi = 0 \;\; on \;\; \Gamma\phi \end{array}$$

$$\Rightarrow \delta J = -\int_{\Omega} \qquad\qquad \delta\phi \, d\Omega + \int_{\Gamma_q} (k\phi_n - \overline{q})\, \delta\phi \, d\Gamma + \int_{\Gamma_f} k\phi_n \,\delta\phi \, d\Gamma = 0$$

$$\delta\phi = Arbitrary$$

$$Euler.\, equ \Rightarrow \qquad\qquad = 0 \qquad in\ \Omega$$

$$k\phi_n - \overline{q} = 0 \quad on \quad L_q$$

$$\delta\phi = 0 \quad or \quad k\phi_n = 0 \quad \Gamma_f$$

Heat Conduction :

If	k & Q	constant	$\nabla^2 \phi =$ const.	Poisson's equation
If	$k = 1$ & $Q = 0$		$\nabla^2 \phi = 0$	Laplace equation

Also other form of equations such as Torsion problem (Poisson's equation) or Irrotational flow (Laplace equation), seepage problem or flow through porous media are examples of the above equations.

Example5. Transient 2-D Heat

Equivalent steady state variational principle for any time t :

$$J(\phi) = T\int_{\Omega} \left[\frac{1}{2} \left\{ k_x \phi_x^2 + k_y \phi_y^2 \right\} - Q\phi + 2C\dot{\phi}\phi \right] dxdy + T\int_{\Gamma_A} \overline{q}_A \,\phi \, d\Gamma + T\int_{\Gamma_C} \left\{ \overline{q}_C + \alpha \left(\frac{\phi}{2} - \overline{\phi}_C \right) \phi \right\} d\Gamma$$

ϕ is a function of x,y and time t $\qquad \dot{\phi} = \dfrac{\partial\phi}{\partial t}$

$\delta J(\phi) = 0$ at any time t

$\dfrac{\partial\phi}{\partial t}$ must be considered fixed in the calculus of variation formulation

$$\delta J(\phi) = T\int_{\Omega}\left[k_x\,\phi_x\,\delta\phi_x + k_y\,\phi_y\,\delta\phi_y - Q\delta\phi + C\delta\dot{\phi}\phi + C\dot{\phi}\,\delta\phi\right]dA$$

$$+\,T\int_{\Gamma_A}\bar{q}_A\,\delta\phi\,d\Gamma + T\int_{\Gamma_C}\left[\bar{q}_C\;\delta\phi + \alpha\,\phi\,\delta\phi - \alpha\,\bar{\phi}_C\,\delta\phi\right]d\Gamma$$

$$= T\int_{\Gamma}k_x\phi_x\nu_x\,\delta\phi\,d\Gamma - \int_{\Omega}(k_x\phi_x)_x\;\delta\phi\,d\Omega + \int_{\Gamma}k_y\phi_y\,\delta\phi\,\nu_y\,d\Gamma - \int_{\Omega}(k_y\phi_y)_y\,\delta\phi\,d\Omega$$

$$-\int Q\,\delta\phi\,d\Omega + \int C\dot{\phi}\;\delta\phi\,d\Omega + T\int_{\Gamma_A}\bar{q}_A\,\delta\phi\,d\Gamma + T\int_{\Gamma_C}(q'_C + \alpha\,\phi - \alpha\,\bar{\phi}_C)\,\delta\phi\,d\Gamma = 0$$

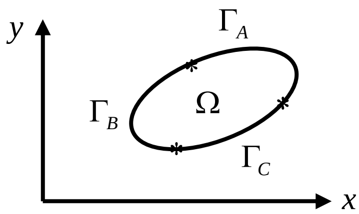
$$(k_x\phi_x)_x + (k_y\phi_y)_y + Q - C\dot{\phi} = 0$$

$$\delta T(\phi) = T\int_{\Omega}\left[-(k_x\phi_x)_x - (k_y\phi_y)_y - Q + C\dot{\phi}\,\right]\delta\phi\,d\Omega$$

$$+\,T\int_{\Gamma}(k_x\phi_x\nu_x + k_y\phi_y\nu_y)\;\delta\phi\,d\Gamma + T\int_{\Gamma_A}\bar{q}_A\;\delta\phi\,d\Gamma$$

$\Gamma = \Gamma_A + \Gamma_B + \Gamma_C$

$$+\,T\int_{\Gamma_C}\bar{q}_C + \alpha\,(\phi - \phi_C)\;\delta\phi\,d\Gamma = 0$$



$\Gamma = \Gamma_A + \Gamma_B + \Gamma_C$

$T = Thickncss$

On Γ_B $\quad \phi = \phi_B$

Γ_A $\quad k_x\phi_x\nu_x + k_y\phi_y\nu_y + \bar{q}_A = 0$

Γ_C $\quad k_x\phi_x\nu_x + k_y\phi_y\nu_y + \bar{q}_C + \alpha\,(\phi - \phi_C) = 0$

$\phi = tempreature$

$k_x = Thermal\ Conductivity\ in\ x\ direction$

$k_y = Thermal\ Conductivity\ in\ y\ direction$

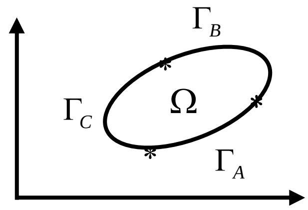
$Q = Heat\ input\ per\ unit\ volume$

$\bar{q}_A\,,\,\bar{q}_C = specified\ heat\ input\ per\ unit\ area$
$on\ \Gamma_A\ and\ \Gamma_C\ respectively$

* Problem: For a transient 2-D heat flow, the equivalent steady state variational principle at time t can be written as:

$$J(\phi) = \int_{\Omega} \frac{1}{2}\left[\left(k_x\phi_x^2 + k_y\phi_y^2\right) - Q\phi + 2C\dot{\phi}\phi\right] dxdy + \int_{\Gamma_A} \overline{q}_A \phi \, d\Gamma + \int_{\Gamma_C} \left\{\overline{q}_C + \alpha\left(\frac{\phi}{2} - \overline{\phi}_C\right)\phi\right\} d\Gamma$$

$$\phi = \int(x, y, t) \qquad \dot{\phi} = \frac{\partial \phi}{\partial t} \qquad k_x \, , k_y \qquad Q \qquad \overline{q}_A \, , \overline{q}$$



$$\Gamma = \Gamma_A + \Gamma_B + \Gamma_C$$

You are asked to find the Euler equation and the appropriate boundary condition

1- assumption about displacement field
2- sometime assumption about constitutive law
3- variational process as it relates to the T.P.E
4- it gives us proper equations of equilibrium and proper BC'S (Certain internal constraints due to displacement assumptions)

8. No. of Rigid body modes in a system

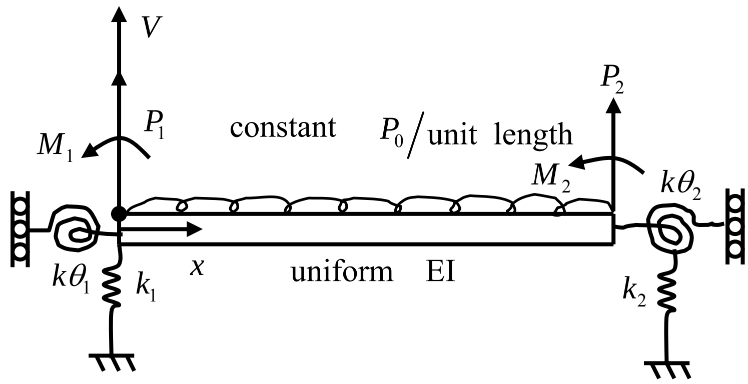
In a variational form we try to find the strain energy U.
The rigid body motions are not accompanied by change in strain energy.
The No. of non contributing terms (from the displacement model) to the strain energy are the No. of rigid body modes.

"Bathe" P.173

If the structure is not supported, there will be a number of linearly independent vectors, $U_1, U_2, \ldots\ldots, V_q$ for which the expression $U_i^T \, K \, U_i$ is equal to zero, i.e. zero strain energy is stored in the system when U_i is the displacement vector. Such vector U_i is said to represent a rigid body mode of the system.

9. Sample Problems

1- For the beam shown, write down the variational principle (Potential Energy) which also includes the boundary actions.
Find out the Euler Lagrange equation and the associated boundary conditions.



K_1, K_2 are translational spring constnts
$K\theta_1, K\theta_2$ are rotational spring constnts

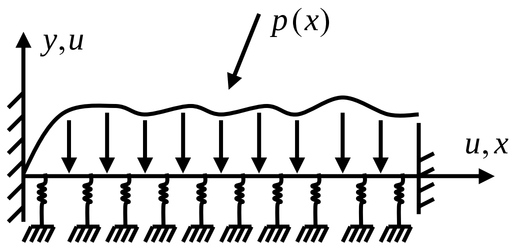
Are there any rigid body modes present?

2- Figure 2 show a system of beam-column with transverse and tangential springs.

a) Write down the functional (Total Potential Energy) for the system. Perform the first variation (Fig 2).

b) Derive the Euler-Lagrange equations and the associated bc's

c) How many rigid body modes do exist?



d) Perform the second variation $\delta^2\Pi$ to show weather the problem is a minimum or maximum.
Note: $\delta^2\Pi = \delta(\delta\Pi)$